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Regularity theory for Hamilton–Jacobi equations

Diogo Aguiar Gomes[a,b,1]

[a] *Department of Mathematics, University of Texas at Austin, RLM 8.100, Austin, TX 78712, USA*
[b] *Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, Lisboa 1096, Portugal*

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Abstract

The objective of this paper is to discuss the regularity of viscosity solutions of time independent Hamilton–Jacobi Equations. We prove analogs of the KAM theorem, show stability of the viscosity solutions and Mather sets under small perturbations of the Hamiltonian.

1. Introduction

The objective of this paper is to study the regularity and stability under small perturbations of viscosity solutions of Hamilton–Jacobi equations

$$H(P + D_x u, x) = \bar{H}(P), \tag{1}$$

using a new set of ideas that combines dynamical systems techniques with control theory and viscosity solutions methods. In (1), $H(p,x): \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth Hamiltonian, strictly convex, i.e., $D^2_{vv}L(x,v) > \gamma > 0$ uniformly (this is also called uniformly convex by some authors), and coercive in p ($\lim_{|p|\to\infty} \frac{H(p,x)}{|p|} = \infty$), and $\mathbb{Z}^n$ periodic in x ($H(p, x+k) = H(p,x)$ for $k \in \mathbb{Z}^n$). Since $\mathbb{R}^n$ is the universal covering of the n-dimensional torus, we identify H with its projection $\mathrm{pr}\, H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$. By changing conveniently the Hamiltonian we may take $P = 0$ and $\bar{H}(P) = \bar{H}$, which we will do throughout the paper to simplify the notation.

E-mail address: dgomes@math.ist.utl.pt.

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In general, (1) does not admit global smooth solutions. The KAM theorem deals with the case in which

$$H(p,x) = H_0(p) + \varepsilon H_1(p,x). \tag{2}$$

Under generic conditions it is possible to prove that for most values of P and sufficiently small ε (1) admits a smooth solution that can be approximated by a power series in ε [Arn89]. In this paper, we will prove analogous results for viscosity solutions of (1).

The outline of this paper is the following: in Section 2 we review basic facts concerning the connections between Mather measures and viscosity solutions. A general reference on control theory and viscosity solutions is [FS93]. The special results concerning viscosity solutions of (1) can be found in [Con95,Con97,LPV88]. The main references on Mather's theory are [Mat89a,Mat89b,Mat91,Mn92,Mn96]. The use of viscosity solutions to study Hamiltonian systems, and in particular Mather's theory is discussed by Fathi [Fat97a,Fat97b,Fat98a,Fat98b,W99] and Jauslin, Kreiss and Moser [JKM99] (for conservation laws in one dimension). Further developments and applications were considered in [EG01,Gom00,Gom01b]. Then we review representation formulas for $\bar{H}$ [CIPP98].

In Section 3 study the behavior of $\bar{H}$ as a function of ε. We prove that $\bar{H}$ is Lipschitz in ε, and depending only on properties of the unperturbed problem, we show that $\bar{H}$ is differentiable with respect to ε. Then we obtain L^2 estimates (with respect to Mather measures) on the differences $D_x u^\varepsilon - D_x u$ (u and u^ε are solutions of (1) for $\varepsilon = 0, \varepsilon$, respectively), as well as some perturbative results for the expansion of u^ε is a power series in ε. Such results are an analog of the KAM theorem for viscosity solutions. In particular they show L^2 stability of the Mather sets.

These estimates are fairly general, and to prove finer results, in Section 4 we assume the additional hypothesis that the Mather measure is uniquely ergodic. The main idea is that, like in KAM theory, a non-resonance-type condition should be imposed to prove stronger stability results. This role is played by unique ergodicity of the Mather measure. We show, in Section 4, that u^ε is uniformly continuous in ε.

2. Mather measures and viscosity solutions

The purpose of this section is to review some results concerning viscosity solutions and Mather measures.

Theorem 1 (Lions, Papanicolaou, and Varadhan). *For each $P \in \mathbb{R}^n$ there exists a number $\bar{H}(P)$ and a periodic viscosity solution u of* (1). *The solution u is Lipschitz, semiconcave, and $\bar{H}$ is a convex function of P.*

Both $\bar{H}$ and the viscosity solutions of (1) encode the dynamics certain trajectories (global minimizers, see [EG01]) of the Hamilton equations

$$\dot{x} = -D_p H(p, x), \qquad \dot{p} = D_x H(p, x). \tag{3}$$

Let L, the Lagrangian, be the Legendre transform of H

$$L(x, v) = \sup_v -pv - H(p, x). \tag{4}$$

This Lagrangian is defined on the tangent space of the torus (or when convenient one considers its lifting to the tangent space $\mathbb{R}^n \times \mathbb{R}^n$ of the universal covering $\mathbb{R}^n$ of $\mathbb{T}^n$). Note that through the paper we use the control theory convention, i.e. (3) is time reversed (in classical mechanics one has $\dot{x} = D_p H(p, x)$ and $\dot{p} = -D_x H(p, x)$) and the Legendre transform (4) also has an extra minus sign (instead of $\sup_v pv - H(p, x)$).

Theorem 2 (Mather). *For each P there exists a positive probability measure μ (Mather measure) on $\mathbb{T}^n \times \mathbb{R}^n$ invariant under the dynamics* (3). *This measure minimizes*

$$\int L(x, v) + Pv \, d\mu$$

over all such measures.

Several important properties of Mather measures can be described in terms of viscosity solutions. Mather measures, as defined in the previous theorem, are supported in the tangent space of the torus—however it is convenient to consider another measure on the cotangent space of the torus induced by μ using the diffeomorphism $v = -D_p H(p, x)$. By abuse of language we will call again Mather measure to such measure.

Theorem 3 (Fathi). *Suppose μ is a Mather measure and let u be any solution of* (1). *Then μ is supported on the graph $(x, P + D_x u)$. Furthermore, $D_x u$ is Lipschitz on the support of μ.*

The fact that the support of a Mather measure is a Lipschitz graph was proven by Mather [Mat89b]. Therefore, once it is known that μ is supported on the graph $(x, P + D_x u)$ the last part of the theorem follows trivially. Similar statements can also be found in [W99] or, using entropy solutions for conservation laws instead of viscosity solutions of Hamilton–Jacobi equations in [JKM99]. The next proposition gives more precise Lipschitz estimates on $D_x u$ and shows that even outside the Mather set $D_x u$ is Lipschitz.

Proposition 1. *Suppose (x,p) is a point in the graph*

$$\mathcal{G} = \{(x, D_x u(x)) : \text{ } u \text{ is differentiable at } x\}.$$

Then for all $t>0$ the solution $(x(t), p(t))$ of (3) *with initial conditions (x,p) belongs $\mathcal{G}$. If for some $T>0$, $(x(-T), p(-T)) \in \mathcal{G}$ then for any y such that $D_x u(y)$ exists*

$$|D_x u(x) - D_x u(y)| \leqslant C|x-y|$$

with a constant depending on T.

Proof. The first part of the theorem (invariance of the graph for $t>0$) is a consequence of the optimal control interpretation of viscosity solutions [FS93] and the reader may find a proof, for instance in [Gom01b] or [Gom00]. To prove the second part, let S be the set of the points x such that $D_x u$ exists and the solution of (3) with initial conditions $(x, D_x u)$ stays in $\mathcal{G}$ up to time $t=-T<0$. We claim that

$$|u(x+y) - 2u(x) + u(x-y)| \leqslant C(T)|y|^2$$

for all $x \in S$ and all $y \in \mathbb{R}^n$. Given this claim, the result follows from the proof in [EG01], Section 6. Part of the claim

$$u(x+y) - 2u(x) + u(x-y) \leqslant C|y|^2$$

is just a consequence of semiconcavity of viscosity solutions, and the constant C does not depend on T [FS93]. Thus it suffices to prove

$$u(x+y) - 2u(x) + u(x-y) \geqslant -C|y|^2.$$

Let $x(s)$, $0 \leqslant s \leqslant T$, be a solution of (3). Set $z = x(0)$, $x = x(T)$. Observe that

$$u(z) = \int_0^T L(x(s), \dot{x}(s)) + \bar{H}\, ds + u(x)$$

and for any ψ

$$u(z) \leqslant \int_0^T L(x(s) + \psi, \dot{x}(s) + \dot{\psi}(s)) + \bar{H}\, ds + u(x + \psi(T)).$$

Choose $\psi(s) = \pm \frac{y}{T s}$ to get

$$u(x+y) + u(x-y) - 2u(x) \geqslant -C(T)|y|^2. \qquad \square$$

Note that $C(T) = O(\frac{1}{T})$, as $T \to 0$. Simple examples show that this is sharp—as one would expect $D_x u$ is not globally Lipschitz and the Lipschitz constant depends on "how much time it takes to hit a shock".

Let ϕ be a Lipschitz function. We need to define what $D_x\phi(x)$ means in the support of a Mather measure. The problem is that although ϕ is differentiable almost everywhere with respect to Lebesgue measure, a measure μ may be supported exactly where the derivative does not exist. However, there is a natural definition of derivative that is convenient for our purposes.

A function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a *version* of $D_x\phi$ if the graph of ψ is contained in the vertical convex hull of the closure of the graph of $D_x\phi$. More precisely if

$$\psi(x) \in \mathscr{D}_x\phi(x),$$

where

$$\mathscr{D}_x\phi(x) = \mathrm{co}\left\{ p : p = \lim_{n\to\infty} D_x\phi(x_n) \text{ with } x_n \to x, \phi \text{ differentiable at } x_n \right\}.$$

The next two propositions show that this definition is quite natural and useful to our purposes:

Proposition 2. *Assume that ϕ has the property that if $x_n \to x$ and ϕ is differentiable at x and at each x_n then $D_x\phi(x_n) \to D_x\phi(x)$. Then any version of $D_x\phi$ coincides with the derivative of ϕ at all points where ϕ is differentiable.*

Proof. The hypothesis on ϕ implies immediately that

$$\mathscr{D}_x\phi(x) = \{D_x\phi(x)\}$$

if ϕ is differentiable at x. $\square$

The solutions of (1) have this property but this is not true for general Lipschitz functions.

Proposition 3. *Suppose (x,p) is a point in the graph $\mathscr{G}$. Let $(x(t),p(t))$ be a solution of* (3) *with initial conditions (x,p) If for some $T>0$, $(x(T),p(T)) \in \mathscr{G}$ then for any y and any version $D_xu(y)$*

$$|D_xu(x) - D_xu(y)| \leqslant C|x-y|$$

with a constant depending on T.

Proof. This follows from Proposition 1 and from observing that $|\cdot|$ is a convex function. $\square$

Since Mather measures μ are invariant under dynamics (3) one has for smooth functions ϕ

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_x\phi(y)D_pH(p,y)\,d\mu = 0.$$

We prove next that for Lipschitz functions ϕ it is possible to choose a version of $D_x\phi$ such that the same identity holds.

Theorem 4. *Let $\phi:\mathbb{R}^n\to\mathbb{R}$ be a Lipschitz function and μ a Mather measure. Then there exists a version of $D_x\phi$ such that*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} D_x\phi(y)D_pH(p,y)\,d\mu = 0.$$

Proof. Consider a generic point (x,p) in the support of μ and the corresponding trajectory $(x(t),p(t))$ of (3) with initial condition (x,p). Let T_n be a sequence converging to $+\infty$. Through some subsequence

$$\frac{1}{T_n}\int_0^{T_n}\varphi(p(t),x(t))\,dt\to\int_{\mathbb{T}^n\times\mathbb{R}^n}\varphi\,d\mu$$

for all μ-integrable, continuous, and periodic (in x) functions φ. Let $z_n\in\mathbb{R}^n$ be any sequence such that $|z_n|\to 0$. If φ is continuous and does not depend on p then

$$\frac{1}{T_n}\int_0^{T_n}\varphi(z_n+x(t))\,dt\to\int_{\mathbb{T}^n\times\mathbb{R}^n}\varphi\,d\mu.$$

Let ϕ be a Lipschitz function. Note that ϕ is differentiable almost everywhere. Thus it is possible to choose $z_n\to 0$ such that, for each n, $D_x\phi(z_n+x(t))$ is defined for almost every t. Now consider the sequence of vector-valued measures η_n defined by

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\zeta(p,y)\,d\eta_n=\frac{1}{T_n}\int_0^{T_n}D_x\phi(z_n+x(t))\,\zeta(p(t),x(t))\,dt$$

for all vector valued smooth, and periodic in y, functions ζ. Since $D_x\phi$ is bounded, we can extract subsequence, also denoted by η_n, that converges weakly to a vector measure η.

Since $\eta\ll\mu$, in the sense that for any set A, $\mu(A)=0$ implies that the vector $\eta(A)=0$. Therefore, by Radon–Nikodym theorem, we have $d\eta=\psi d\mu$, for some $L^1(\mu)$ function ψ. By standard techniques in weak limits it is clear that for almost every $x\in\mathbb{T}^n$ the density ψ is in $\mathscr{D}_x\phi$, so it is a version of $D_x\phi$.

Finally, to see that

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\psi D_pH(p,y)\,d\mu=0,$$

we just have to observe that

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H(p, y) d\eta_n = O(\varepsilon_n)$$

and so

$$0 = \int_{\mathbb{T}^n \times \mathbb{R}^n} D_p H(p, y) d\eta = \int_{\mathbb{T}^n \times \mathbb{R}^n} \psi D_p H(p, y) d\mu. \qquad \square$$

The Hamilton–Jacobi equation (1) has two unknowns $\bar{H}$ and u. In the remaining of this section, we recall some representation formulas for $\bar{H}$ that do not involve solving (1). A classical result [LPV88] is that

$$\bar{H} = -\lim_{\alpha \to 0} \inf_{x(\cdot)} \alpha \int_0^\infty L(x, \dot{x}) e^{-\alpha t}\, dt$$

with the infimum taken over all Lipschitz trajectories $x(\cdot)$. There are two distinct formulas more convenient for our purposes—both will be optimization problems—the first one, which makes a connection between Mather's problem and viscosity solutions, is

$$\bar{H} = -\inf_{\mu} \int L\, d\mu \tag{5}$$

in which the measure μ is a generalized curve, i.e.

$$\int v D_x \phi\, d\mu = 0$$

for all smooth ϕ. This expression for $\bar{H}$ has a dual formula that consists in an L^∞ calculus of variations problem. This result was first proven in [CIPP98]. In [Gom00] it was proved using Legendre–Fenchel duality theory, and stochastic generalization was studied in [Gom01a].

Theorem 5 (Contreras, Iturriaga, Paternain, and Paternain).

$$\bar{H} = \inf_{\phi} \sup_{x} H(D_x \phi, x), \tag{6}$$

where the infimum is taken over all periodic smooth functions ϕ.

Unfortunately, this representation formula for $\bar{H}$ does not yield a method to compute the viscosity solution u. A sequence of minimizers u_n may or may not converge to a viscosity solution of (1).

Now we discuss the Euler–Lagrange equations for this problem.

Proposition 4. *Suppose u is a smooth solution of* (1) (*and therefore is a minimizer of* (6)). *Then*

$$\sup_x D_pH(D_xu, x)D_x\phi \geqslant 0 \tag{7}$$

for all smooth and periodic ϕ.

Proof. Assume u solves (1) and therefore is a minimizer of (6). Then for any ϕ smooth and periodic

$$H(D_xu + \varepsilon D_x\phi, x) \leqslant H(D_xu, x) + \varepsilon D_pH(D_xu, x)D_x\phi + O(\varepsilon^2).$$

Therefore,

$$\sup_x H(D_xu + \varepsilon D_x\phi, x) \leqslant \bar{H} + \varepsilon \sup_x D_pH(D_xu, x)D_x\phi + O(\varepsilon^2).$$

Since

$$\sup_x H(D_xu + D_x\phi, x) \geqslant \bar{H}$$

we must have

$$\sup_x D_pH(D_xu, x)D_x\phi \geqslant 0$$

for any ϕ smooth and periodic. $\square$

3. L^2-Perturbation theory

In this section, we assume the Hamiltonian to be

$$H(p, x; \varepsilon) = H_0(p, x) + \varepsilon H_1(p, x)$$

as in (2). We assume that ε is always sufficiently small such that $H(p, x; \varepsilon)$ is strictly convex in p. The main objective is to obtain estimates that show that the solution of the perturbed problem ($\varepsilon \neq 0$) is close to the unperturbed problem ($\varepsilon = 0$). In particular, we prove that under appropriate hypothesis

$$\int |D_xu^\varepsilon - D_xu^0|^2 \, d\nu \leqslant C\varepsilon^2$$

in which u^ε and u^0 are solutions of (1) and ν is a Mather measure. Then using the similar techniques, we prove estimates on approximate solutions using an iterative procedure. In spirit, this is close to the KAM theory in which a solution of (1) is obtained as a power series. However, because viscosity solution theory guarantees the existence of a solution of (1) for any ε (as long as the Hamiltonian is strictly convex) one can show that such a series is asymptotic to the solution without having to worry about convergence or existence of a solution.

We proceed as follows: first we study the dependence on ε of $\bar{H}_\varepsilon$. Then we show that differentiability properties of $\bar{H}_\varepsilon$ characterize L^2 properties of the viscosity solutions. More precisely, twice differentiability in ε of $\bar{H}_\varepsilon$ implies D_xu^ε is L^2 close (with respect to a Mather measure) to $D_x u$. Finally, we consider certain asymptotic approximations and prove L^2 estimates between the solution and approximate solutions of

$$H(P + D_x u^\varepsilon, x; \varepsilon) = \bar{H}_\varepsilon(P). \tag{8}$$

Proposition 5. *Suppose $H(p, x; \varepsilon) = H_0(p, x) + \varepsilon H_1(p, x)$ with H_0 strictly convex in p and H_1 bounded with bounded derivatives. Then for each P and ε sufficiently small there exists a unique $\bar{H}_\varepsilon(P)$ and a viscosity solution u^ε of* (8). *Furthermore, the function $\bar{H}_\varepsilon(P)$ is convex in P and Lipschitz in ε.*

Proof. The existence of $\bar{H}_\varepsilon(P)$ as well as convexity in P follows from the results in [LPV88]. Thus, it suffices to prove the Lipschitz property. Observe that

$$|\bar{H}_{\varepsilon_1} - \bar{H}_{\varepsilon_2}| \leqslant |\varepsilon_1 - \varepsilon_2| \sup_{|p| \leqslant R} \sup_x |H_1(p, x)|$$

with R being an upper bound on the Lipschitz constant for the viscosity solutions of (1). $\square$

An interesting observation is that if $H_1(p, x) = V(x)$ (no dependence on p) then $\bar{H}$ is a convex function of ε. To see this note that

$$L(x, v) = L_0(x, v) - \varepsilon V(x)$$

(L_0 is the Legendre transform of H_0) and from (5)

$$\bar{H} = \sup_\mu - \int L \, d\mu$$

in which the supremum is taken over all probability measures, invariant under (3). Since $-L$ is a convex function of ε, and the supremum of convex functions is convex, $\bar{H}$ is convex in ε and therefore twice differentiable in ε almost everywhere.

In the next theorem, we compute an expansion of $\bar{H}_\varepsilon$ close to $\varepsilon = 0$ in terms of Mather measures and viscosity solutions. In Theorem 8 we will show that such an expansion implies that regularity of $\bar{H}_\varepsilon$ yields regularity for the viscosity solutions.

Theorem 6. *Let μ be a Mather measure corresponding to the unperturbed problem ($\varepsilon = 0$) and ν its projection in the x coordinates. Then*

$$\bar{H}_\varepsilon \geqslant \bar{H}_0 + \varepsilon \bar{H}_1 + \gamma \int |D_x u^\varepsilon - D_x u|^2 d\nu + O(\varepsilon^2) \tag{9}$$

in which

$$\bar{H}_1 = \int H_1(D_x u, x) \, d\nu,$$

u and u^ε are viscosity solutions of (1) *for $\varepsilon = 0, \varepsilon$ and $D_x u^\varepsilon$ denotes a version of $D_x u^\varepsilon$.*

Proof. Observe that for any version of $D_x u^\varepsilon$

$$\bar{H}_\varepsilon \geqslant H_0(D_x u^\varepsilon, x) + \varepsilon H_1(D_x u^\varepsilon, x)$$

and so by strict convexity

$$\bar{H}_\varepsilon \geqslant \bar{H}_0 + \varepsilon H_1(D_x u, x) + D_p H_0(D_x u^\varepsilon - D_x u) + \gamma |D_x u^\varepsilon - D_x u|^2 + O(\varepsilon^2).$$

Integrate with respect to $d\nu$ and use the fact that

$$\int D_p H_0(D_x u^\varepsilon - D_x u)\, d\nu = 0$$

to get

$$\bar{H}_\varepsilon \geqslant \bar{H}_0 + \varepsilon \bar{H}_1 + \gamma \int |D_x u^\varepsilon - D_x u|^2 d\nu + O(\varepsilon^2). \qquad \square$$

This theorem implies that $\bar{H}_\varepsilon$ has always non-empty subdifferential at $\varepsilon = 0$ ($\bar{H}_\varepsilon \geqslant \bar{H}_0 + \varepsilon \bar{H}_1 + O(\varepsilon^2)$). Therefore, if $\bar{H}_\varepsilon$ is differentiable at $\varepsilon = 0$ its derivative is $\bar{H}_1$. Next we discuss a converse inequality and prove that under suitable conditions $\bar{H}_\varepsilon = \bar{H}_0 + \varepsilon \bar{H}_1 + O(\varepsilon^2)$, and therefore $\bar{H}_\varepsilon$ is differentiable at $\varepsilon = 0$.

Theorem 7. *Assume u is a smooth solution of the unperturbed problem corresponding to $\varepsilon = 0$. Let ν be, as in the previous theorem, the projection of a Mather measure corresponding to $\varepsilon = 0$. Suppose there exists a smooth function v and a number $\bar{H}_1$ such that*

$$D_p H_0(D_x u, x) D_x v + H_1(D_x u, x) = \bar{H}_1. \tag{10}$$

Then

$$\bar{H}_1 = \int H_1(D_x u, x)\, d\nu \tag{11}$$

and

$$\bar{H}_\varepsilon \leqslant \bar{H}_0 + \varepsilon \bar{H}_1 + O(\varepsilon^2). \tag{12}$$

Proof. Let ν be the x projection of a Mather measure corresponding to the unperturbed problem ($\varepsilon = 0$). First observe that (10) implies (11) simply by integration with respect to $d\nu$. Recall that

$$\bar{H}_\varepsilon \leqslant \sup_x \; H_0(D_x u + \varepsilon D_x v, x) + \varepsilon H_1(D_x u + \varepsilon D_x v, x).$$

Since

$$H_0(D_x u + \varepsilon D_x v, x) + \varepsilon H_1(D_x u + \varepsilon D_x v, x) = \bar{H}_0 + \varepsilon \bar{H}_1 + O(\varepsilon^2),$$

it follows

$$\bar{H}_\varepsilon \leqslant \bar{H}_0 + \varepsilon \bar{H}_1 + O(\varepsilon^2),$$

as claimed. $\square$

The next step in our program is to show that regularity of $\bar{H}_\varepsilon$ actually implies regularity for the viscosity solutions u^ε.

Theorem 8. *Suppose $\bar{H}_\varepsilon$ twice differentiable in ε. Then, for any Mather measure μ (and corresponding projection ν) there exists a version of $D_x u^\varepsilon$ such that*

$$\int |D_x u^\varepsilon - D_x u|^2 \, d\nu \leqslant C\varepsilon^2.$$

Proof. Observe that for any version of $D_x u^\varepsilon$

$$\begin{aligned}\bar{H}_\varepsilon &\geqslant H_\varepsilon(D_x u^\varepsilon, x)\\ &\geqslant H_0(D_x u, x) + D_p H_0(D_x u, x)(D_x u^\varepsilon - D_x u) + \gamma |D_x u^\varepsilon - D_x u|^2\\ &\quad + \varepsilon H_1(D_x u, x) - C\varepsilon |D_x u^\varepsilon - D_x u|.\end{aligned}$$

Integrating with respect to the projection ν and using Theorem 4.

$$\bar{H}_\varepsilon - \bar{H}_0 - \varepsilon \bar{H}_1 + O(\varepsilon^2) \geqslant \frac{\gamma}{2} \int |D_x u^\varepsilon - D_x u|^2 d\nu.$$

Since $\bar{H}_\varepsilon$ is twice differentiable in ε (the remark after Theorem 6 implies that $D_\varepsilon \bar{H}_\varepsilon = \bar{H}_1$ at $\varepsilon = 0$) we conclude

$$\int |D_x u^\varepsilon - D_x u|^2 d\nu \leqslant C\varepsilon^2. \qquad \square$$

An alternate way to state the previous theorem is that for any y sufficiently small (for instance $|y| \leqslant \varepsilon^2$) we have

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T |D_x u^\varepsilon(x(t) + y) - D_x u(x(t))|^2 dt \leqslant C\varepsilon^2, \tag{13}$$

provided $\bar{H}_\varepsilon$ is twice differentiable at $\varepsilon = 0$ and $x(t)$ is a orbit of (3) for $\varepsilon = 0$ with initial conditions on the Mather set.

The remaining part of this section is dedicated to the study of high-order methods. The idea is that given an integer $n \geqslant 0$, by solving a hierarchy of equations, one can compute a function $\tilde{u}^\varepsilon$ such that

$$\begin{aligned}&H_0(D_x \tilde{u}^\varepsilon, x) + \varepsilon H_1(D_x \tilde{u}^\varepsilon, x)\\ &\qquad = \bar{H}_0 + \varepsilon \bar{H}_1 + \cdots + \varepsilon^{n-1} \bar{H}_{n-1} + O(\varepsilon^n).\end{aligned} \tag{14}$$

We call such a function an approximate solution of order n. To compute $\tilde{u}^\varepsilon$ write

$$\tilde{u}^\varepsilon = u + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \ .$$

The first equation is

$$H_0(D_xu, x) = \bar{H}_0,$$

the second is

$$D_pH_0(D_xu, x)v_1 + H_1(D_xu, x) = \bar{H}_1$$

with $\bar{H}_1 = \int H_1(D_xu, x)\, d\nu$, the remaining equations are

$$D_pH_0(D_xu, x)v_k + f_k(D_xu, D_xv_1, \ldots, D_xv_{k-1}, x) = \bar{H}_k$$

with $\bar{H}_k = \int f_k(D_xu, D_xv_1, \ldots, D_xv_{k-1}, x)\, d\nu$, here f_k is some function that can be computed by assembling together the remaining terms of order ε^k. Assuming that such equations can be solved we have immediately

$$\bar{H}_\varepsilon \leqslant \bar{H}_0 + \varepsilon\bar{H}_1 + \cdots + \varepsilon^{n-1}\bar{H}_{n-1} + O(\varepsilon^n) \tag{15}$$

as in Theorem 7.

Let $\tilde{\nu}$ be a measure defined by

$$\int \phi\, d\tilde{\nu} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(x(t))\, dt$$

in which $\dot{x}(t) = D_pH_0(D_x\tilde{u}^\varepsilon, x) + \varepsilon D_pH_1(D_x\tilde{u}^\varepsilon, x)$ and ϕ is any continuous periodic function. We call $\tilde{\nu}$ an approximate Mather measure. Note that if φ is smooth and periodic then

$$\int D_x\varphi[D_pH_0(D_x\tilde{u}^\varepsilon, x) + \varepsilon D_pH_1(D_x\tilde{u}^\varepsilon, x)]\, d\tilde{\nu} = 0, \tag{16}$$

and as before, if φ is Lipschitz then (16) holds for a version of $D_x\varphi$.

Let u^ε be a solution of (1). Our objective is to estimate $D_xu^\varepsilon - D_x\tilde{u}^\varepsilon$.

Theorem 9. *Let u^ε be a solution of* (1) *and $\tilde{u}^\varepsilon$ an approximate solution of order n. Then there exists a version of D_xu^ε such that*

$$\int |D_xu^\varepsilon - D_x\tilde{u}^\varepsilon|^2 d\tilde{\nu} \leqslant C\varepsilon^n. \tag{17}$$

Proof. Observe that for any version of D_xu^ε, the strict convexity of H_0 yields

$$\bar{H}_\varepsilon \geqslant H_0(D_xu^\varepsilon, x) + \varepsilon H_1(D_xu^\varepsilon, x).$$

Thus

$$\begin{aligned}\bar{H}_\varepsilon \geqslant\ & H_0(D_x\tilde{u}^\varepsilon, x) + \varepsilon H_1(D_x\tilde{u}^\varepsilon, x)\\ &+ [D_pH_0(D_x\tilde{u}^\varepsilon, x) + \varepsilon D_pH_1(D_x\tilde{u}^\varepsilon, x)](D_xu^\varepsilon - D_x\tilde{u}^\varepsilon)\\ &+ \frac{\gamma}{2}|D_xu^\varepsilon - D_x\tilde{u}^\varepsilon|^2.\end{aligned} \tag{18}$$

Integrate with respect to $\tilde{\nu}$ and use the fact that

$$\int [D_pH_0(D_x\tilde{u}^\varepsilon, x) + \varepsilon D_pH_1(D_x\tilde{u}^\varepsilon, x)](D_xu^\varepsilon - D_x\tilde{u}^\varepsilon)\, d\tilde{\nu} = 0.$$

Then using (14) we get

$$\bar{H}_\varepsilon + O(\varepsilon^n) \geqslant \bar{H}_0 + \varepsilon \bar{H}_1 + \cdots + \varepsilon^{n-1}\bar{H}_{n-1} + \frac{\gamma}{2}\int |D_x u^\varepsilon - D_x \tilde{u}^\varepsilon|^2 d\tilde{\nu}.$$

But then (15) implies (17). $\square$

4. Uniform continuity

The results on the previous section show that viscosity solutions of (1) have some degree of regularity in ε. This apparently contradicts the examples in which (1) does not have a unique solution (for fixed ε). Obviously, adding any constant to a viscosity solution of (1) produces another viscosity solution. Furthermore, we know that even up to constants the viscosity solutions are not unique. It is therefore surprising that, under certain general conditions, we can prove that

$$u^\varepsilon(x) \to u(x)$$

uniformly on the support of an uniquely ergodic Mather measure (provided an appropriate constant is added to u^ε). This in particular implies uniqueness of solution on each uniquely ergodic component of the support of a Mather measure.

Proposition 6. *Suppose μ is a Mather measure and ν its projection. Let $\varepsilon_n \to 0$. Then there exists a point x in the support of ν and a corresponding optimal trajectory $x^*(t)$ such that for any T*

$$\sup_{0 \leqslant t \leqslant T} |u(x^*(t)) - u^{\varepsilon_n}(x^*(t))| \to 0$$

as $n \to \infty$, provided $u^{\varepsilon_n}(x) = u(x)$.

Proof. We start by proving an auxiliary lemma.

Lemma 1. *There exist a point (x,p) in the support of μ, and sequences $x_n, \tilde{x}_n \to x$, $p_n, \tilde{p}_n \to p$, with $(x_n, p_n) \in \operatorname{supp} \mu$ optimal pair for $\varepsilon = 0$, and $(\tilde{x}_n, \tilde{p}_n)$ optimal pairs for $\varepsilon = \varepsilon_n$.*

Remark. The non-trivial point of the lemma is that the limits of p_n and $\tilde{p}_n$ are the same.

Proof. Take a generic point (x_0, p_0) in the support of μ. Let $x^*(t)$ be the optimal trajectory for $\varepsilon = 0$ with initial condition (x_0, p_0). Then for all $t > 0$

$$H_0(D_x u(x^*(t)), x^*(t)) = \bar{H}_0.$$

Also, for almost every y

$$H(D_x u^{\varepsilon_n}(x^*(t) + y), x^*(t)) = \bar{H}_{\varepsilon_n} + O(|y|)$$

for almost every t. Choose y_n with $|y_n| \leqslant \varepsilon_n$ such that the previous identity holds. By strict convexity of H in p and Lipschitz continuity of $\bar{H}_\varepsilon$ in ε

$$\dot{x}^*(t)\xi + \theta|\xi|^2 \leqslant C|\varepsilon_n|,$$

where

$$\xi = [D_x u(x^*(t)) - D_x u^{\varepsilon_n}(x^*(t) + y_n)],$$

$$\dot{x}^*(t) = -D_p H_0(D_x u(x^*(t)), x^*(t)),$$

and $\theta > 0$. Note that

$$\left|\frac{1}{T}\int_0^T \dot{x}^*(t)\xi\right| \leqslant \frac{|u(x^*(0)) - u(x^*(T))|}{T} + \frac{|u^{\varepsilon_n}(x^*(0) + y_n) - u^{\varepsilon_n}(x^*(T) + y_n)|}{T}.$$

Therefore, we may choose T (depending on n) such that

$$\left|\frac{1}{T}\int_0^T \dot{x}^*(t)\xi\right| \leqslant \varepsilon_n.$$

Thus

$$\frac{1}{T}\int_0^T |D_x u(x^*(t)) - D_x u^{\varepsilon_n}(x^*(t) + y_n)|^2 \leqslant C\varepsilon_n.$$

Choose $0 \leqslant t_n \leqslant T$ for which

$$|D_x u(x^*(t_n)) - D_x u^{\varepsilon_n}(x^*(t_n) + y_n)|^2 \leqslant C\varepsilon_n.$$

Let $x_n = x^*(t_n)$, $\tilde{x}_n = x^*(t_n) + y_n$, and

$$p_n = P + D_x u(x^*(t_n), P), \quad p_n = D_x u^{\varepsilon_n}(x^*(t_n) + y_n).$$

By extracting a subsequence, if necessary, we may assume $x_n \to x$, $\tilde{x}_n \to x$, etc. $\square$

To see that the lemma implies the proposition, let $x_n^*(t)$ be the optimal trajectory for $\varepsilon = 0$ with initial conditions (x_n, p_n). Similarly, let $\tilde{x}_n^*(t)$ be the optimal trajectory for $\varepsilon = \varepsilon_n$ with initial conditions $(\tilde{x}_n, \tilde{p}_n)$. Then

$$u(x_n) = \int_0^t L_0(x_n^*, \dot{x}_n^*) + \bar{H}_0 \, ds + u(x_n^*(t))$$

and

$$u^{\varepsilon_n}(\tilde{x}_n) = \int_0^t L_{\varepsilon_n}(\tilde{x}_n^*, \dot{\tilde{x}}_n^*) + \bar{H}_{\varepsilon_n} \, ds + u^{\varepsilon_n}(\tilde{x}_n^*(t)).$$

Note that, as $\varepsilon_n \to 0$, $L_{\varepsilon_n} \to L_0$ uniformly on compact sets (here L_ε is the Legendre transform of $H = H_0 + \varepsilon H_1$). On $0 \leqslant t \leqslant T$ both x_n^* and $\tilde{x}_n^*$ converge uniformly, and, since by hypothesis,

$$u(x_n), u^{\varepsilon_n}(\tilde{x}_n) \to u(x),$$

we conclude that

$$u^{\varepsilon_n}(\tilde{x}_n^*(t)) - u(x_n^*(t)) \to 0$$

uniformly on $0 \leqslant t \leqslant T$. Therefore,

$$u^{\varepsilon_n}(x^*(t)) - u(x^*(t)) \to 0$$

uniformly on $0 \leqslant t \leqslant T$. $\square$

Given a viscosity solution u of (1) consider the differential equation

$$\dot{x} = -D_p H(D_x u, x). \tag{19}$$

Given an ergodic Mather measure μ (and respective projection ν) associated with u, (19) restricted to $\mathrm{supp}(\nu)$ defines a flow. We say that the flow (19) is uniquely ergodic if there ν is the unique invariant probability measure with support contained in $\mathrm{supp}(\nu)$.

Theorem 10. *Suppose μ is an ergodic Mather measure associated to a viscosity solution u of* (1) *with $\varepsilon = 0$. Let ν denote the projection on μ. Assume that the flow* (19) *restricted to* $\mathrm{supp}(\nu)$ *is uniquely ergodic. Then*

$$u^{\varepsilon}(x) \to u(x)$$

as $\varepsilon \to 0$, uniformly on the support of ν, provided that an appropriate constant $C(\varepsilon)$ is added to u^{ε}.

Proof. Fix $\kappa > 0$. We need to show that if n is sufficiently large then

$$\sup_{x \in \mathrm{supp}(\nu)} |u^{\varepsilon_n}(x) - u(x)| < \kappa.$$

Choose M such that $||D_x u(x)||, ||D_x u^{\varepsilon_n}(x)|| \leqslant M$. Let $\delta = \frac{\kappa}{8M}$. Cover $\mathrm{supp}\,\nu$ with finitely many balls B_i with radius $\leqslant \delta$. Choose (x,p) as in the previous proposition. Let $(x^*(t), p^*(t))$ be the optimal trajectory for $\varepsilon = 0$ with initial condition (x,p). Then there exists T_δ and $0 \leqslant t_i \leqslant T_\delta$ such that $x_i = x^*(t_i) \in B_i$. Choose n sufficiently large such that

$$\sup_{0 \leqslant t \leqslant T_\delta} |u(x^*(t)) - u^{\varepsilon_n}(x^*(t))| \leqslant \frac{\kappa}{2}.$$

Then, for each y in B_i

$$\begin{aligned} |u(y) - u^{\varepsilon_n}(y)| &\leqslant |u(y) - u(y_i)| + |u(y_i) - u^{\varepsilon_n}(y_i)| \\ &\quad + |u^{\varepsilon_n}(y_i) - u^{\varepsilon_n}(y)| \leqslant 4M\delta + \frac{\kappa}{2} \leqslant \kappa. \qquad \square \end{aligned}$$

Actually, the unique ergodicity hypothesis is not too restrictive since by Mane's results [Mn96] "most" Mather measures are uniquely ergodic (in the sense that after small generic perturbations to the Lagrangian the restricted flow (19) is uniquely ergodic).

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